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Spectral Decomposition for Non-selfadjoint Singular Differential Operators

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1. INTRODUCTION

Let q be a continuous complex-valued function on $[0, \infty)$ or $0 \leqslant x < \infty$. Let $lu = -u'' + qu$ $(' = d/dx)$ for u in $C^{(2)}[0, \infty)$, the complex-valued functions on $[0, \infty)$ with continuous derivatives of order $\leqslant 2$. We define the operator T (or T_q) on $L^2 \equiv L^2[0, \infty)$ as the closure of l restricted to the subspace

$$\{u: u \in C_c^{(2)}[0, \infty), u(0) = 0\}$$

where $C_c^{(2)}$ denotes the functions in $C^{(2)}$ with compact support.

To describe our main result we assume that: (a_1) $\int_0^\infty t\,|q(t)|\,dt < \infty$; (b) A, $\tilde{A}$ are bounded away from 0 on $0 \leqslant s < \infty$ ($A, \tilde{A}$ defined in Section 2). Let w and W be as in Section 2. In particular, $w(s, \cdot)$ is a solution of $-u'' + qu = s^2 u$ on $0 \leqslant x < \infty$ with $w(s, 0) = 0$ for $0 \leqslant s < \infty$. We define S and $S\check{}$ by

$$Sf(s) = \int_0^\infty w(s, x)\, f(x)\, dx$$

$$S\check{}\, g(x) = \int_0^\infty w(s, x)\, W(s)\, g(s)\, ds$$

for f, g continuous with compact support on $[0, \infty)$; and, we let S and $S\check{}$ denote the extensions to L^2 when S or $S\check{}$ is a bounded linear transformation from L^2 into L^2. The motivation for W is given in Section 8. In case $q \equiv 0$,

$$Sf(x) = 2i \int_0^\infty \sin(sx) f(x)\, dx.$$

Let σ_0 denote $\sigma(T)$, the spectrum of T, with points of $[0, \infty)$ removed. By Lemma 2.5, σ_0 is finite. Let P denote the spectral projection corresponding to

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T and σ_0 (see Sect. 3.9 in Chap. VII of [2]). Define the operator M on L^2 by $Mg(s) = s^2 g(s)$, if $g(s)$ and $s^2 g(s) \in L^2$. Let $H_0 = PL^2$ and $H_1 = (I-P)L^2$. Then H_0, H_1 are T invariant, H_0 is finite dimensional and the one theorem of this paper states that $ST_1 = MS$ where T_1 is T restricted to H_1; these three results are stated in more detail in Section 3.

In [5] Naimark gives an "eigenfunction" expansion for the Green's function corresponding to T assuming that $t^2 |q(t)|$ is integrable and (b). Kato [4] shows that if

$$\text{(K)} \qquad \int_0^\infty t\,|q(t)|\,dt < 1,$$

then T is similar to M; the scattering method is used and the similarity transform is not explicitly given. We cannot get Kato's theorem from ours. First, it is not clear from our methods that (K) implies σ_0 is empty. Second, we do not know if (K) implies (b). With the assumptions that $t^2 |q(t)|$ is integrable and (b), the theorem on page 2396 of [3] gives our conclusions in slightly different forms. In all three references above, the proofs concentrate on the resolvent whereas our proof on the similarity transformation S.

We now briefly describe the arrangement of the paper. The basic notation and definitions are given in Sections 1 and 2. Section 2 also contains some lemmas concerning properties of $\sigma(T)$. In Section 3 we prove the theorem using lemmas developed throughout the paper; this section gives a quick overview. Section 5 contains the basic properties of the function v. In Section 6 we show that S, $S^{\check{}}$ are well defined and bounded when q is in Q_δ, which is defined in Section 2; the proof shows that S differs from the Fourier transform by a Hilbert Schmidt operator, in a sense. In Section 7 we show that $ST = MS$ assuming (a_1), (b) and that q is bounded; we remove the latter condition with Lemma 10.2 in Section 10. As a preliminary to proving that $(S \mid H_1)^{-1} = S^{\check{}}$, we show in Section 8 that $S^{\check{}} S = I - P$ for $q(x) = O(e^{-\varepsilon_1 x})$. The proof involves approximating T by a corresponding differential operator on $L^2[0, b]$ and letting $b \to \infty$. In Section 9 we show that S, $S^{\check{}}$ and P depend continuously on q in a sense and then show that $S^{\check{}} = I - P$ if q satisfies (a_1) and (b), by using the corresponding identity of Section 8 and taking limit with respect to q. In Section 10 for q satisfying (a_1) and (b), we show that $ST = MS$, removing the restriction in Section 7, and that $S: H_1 \to L^2$ is one to one, onto and bicontinuous with inverse $= S^{\check{}}$; these two facts are the basis of the proof of the theorem given in Section 3.

2. Notation and Basics

The purpose of this section is to establish notation used throughout the paper and to state some basic facts. Recall that in Section 1 we have already defined L^2, l, T, M, σ_0, P, H_0, H_1, S, $S^{\smile}$. We note that all these quantities except L^2 and M depend on q and that we often will vary q.

Throughout the paper $q(x)$ denotes a continuous complex-valued function on $0 \leqslant x < \infty$ or $[0, \infty)$. In various places in the paper, we will make one of the following three assumptions on q: (a_0) $\int |q(t)|\, dt < \infty$; (a_1) $\int t\,|q(t)|\, dt < \infty$; (a_2) $|q(t)| \leqslant Ce^{-\varepsilon_1 t}$, $t \geqslant 0$, for some C, $\varepsilon_1 > 0$. Here and throughout the paper $\int$ denotes $\int_0^\infty$. Q will denote all q which satisfy (a_1) and $|q|_Q$ will denote $\int (1+t)\,|q(t)|\, dt$ for $q \in Q$. We will often require that q satisfies the "indirect" assumption (b), which is described below. Let $C^{(2)}0, \infty)$ denote the complex-valued functions on $[0, \infty)$ which are twice continuously differentiable. Let $lu = -u'' + qu$ for u in $C^{(2)}[0, \infty)$. We introduce the differential equation

$$\text{(E)} \qquad -u'' + qu = \lambda u$$

where λ is an arbitrary complex number. We relate s and λ be $\lambda = s^2$. Two functions $a(x)$ and $b(x)$ are called asymptotic, denoted $a \sim b$, as $x \to \infty$ if $a(x)/b(x) \to 1$ as $x \to \infty$. Basic to the paper is the existence and properties of the solutions $y(s,x)$ and $\tilde{y}(s,x)$ of (E) which are asymptotic to e^{isx} and e^{-isx}, respectively, as $x \to \infty$ when various assumptions are made on q. To get at y, for example, we express $y(s,x)$ as $e^{isx}v(s,x)$; then by using variation of parameters and formal manipulations, we obtain the following integral equations with unknown functions $v(s,x)$:

$$v(s,x) = 1 + \int_x^\infty [(e^{i2s(t-x)} - 1)/i2s]\, q(t)\, v(s,t)\, dt, \qquad \operatorname{Im} s \geqslant 0; \quad (2.1)$$

$$v(s,x) = 1 + (i/2s) \int_0^x e^{i2s(t-x)} q(t)\, v(s,t)\, dt$$
$$- (i/2s) \int_x^\infty q(t)\, v(s,t)\, dt, \qquad \operatorname{Im} s < 0. \quad (2.2)$$

If we express $\tilde{y}$ as $e^{-isx}\tilde{v}(s,x)$, then as above we obtain the integral equation

$$\tilde{v}(s,x) = 1 - \int_x^\infty [(e^{-i2s(t-x)} - 1)/2si]\, q(t)\, \tilde{v}(s,t)\, dt, \qquad \operatorname{Im} s \leqslant 0, \quad (2.3)$$

with unknown function $\tilde{v}$ and a second integral equation similar to (2.2) for $\operatorname{Im} s > 0$, which we omit.

2.1. LEMMA. *Suppose q satisfies* (a_0). *If $s \neq 0$ and* $\operatorname{Im} s \geqslant 0$ ($\operatorname{Im} s < 0$), *then* (2.1) ((2.2)) *has a unique solution $v(s, \cdot)$ and $y(s,x) = e^{isx}v(s,x)$ is a solution of* (E) *and $y(s,x) \sim e^{isx}$ as $x \to \infty$. Similarly for $\tilde{y}$ and $\tilde{v}$.*

This lemma is proved in [5]. It is proved by noting that if $s \neq 0$, then the integral operator in (2.1) or (2.2), operating on bounded continuous functions with the uniform norms, has norm <1 for functions on $[b, \infty)$ and for b sufficiently large. So, $y = e^{isx}v$ is uniquely defined on $[b, \infty)$, $y \sim e^{isx}$ as $x \to \infty$ by the integral equation and y solves (E) because we can reverse the steps that led to the integral equations. Finally, y, and thus v, are defined on $[0, b)$ since y is a solution of (E).

Because of the central role of the function v, we give v a formal definition.

2.2. DEFINITION. If q satisfies (a_0) and $s \neq 0$, then we define $v(s, x)$ as the unique solution of (2.1) for $\operatorname{Im} s \geqslant 0$ and the unique solution of (2.2) for $\operatorname{Im} s < 0$. It q satisfies (a_1), then for $s = 0$ we define $v(0, x)$ as the unique solution of (2.1). (By Lemmas 2.1 and 5.3, v is well defined in both cases.) $\tilde{v}$ is defined in a similar manner.

The properties of v that are needed are developed in Section 5. In particular, if q satisfies (a_2), then Lemma 5.10 shows that (2.1) has a solution for $-\varepsilon_2 < \operatorname{Im} 2s < 0$ which agrees with the solution of (2.2).

We define $A(s) = v(s, 0)$, $\tilde{A}(s) = \tilde{v}(s, 0)$, $y(s, x) = e^{isx}v(s, x)$, $\tilde{y}(s, x) = e^{isx}\tilde{v}(s, x)$ and $w(s, x) = \tilde{A}(s)\, y(s, x) - A(s)\, \tilde{y}(s, x)$ whenever $v(s, x)$ and $\tilde{v}(s, x)$ are defined. Note that $w(s, \cdot)$ is a solution of (E), $w(s, 0) = 0$ and $w(s, x) \sim \tilde{A}e^{isx} - Ae^{-isx}$ as $x \to \infty$. Note that by Lemma 5.9, if q satisfies (a_1), then $w(0, \cdot) \equiv 0$. In the paper we often assume that for a q which satisfies (a_0), q also satisfies

$$\text{(b)} \qquad |A(s)|, |\tilde{A}(s)| \geqslant \delta, \qquad \text{some} \quad \delta > 0.$$

We let Q_δ denote all q in Q which satisfy (b). Let $W(s) \equiv -1/2\pi A(s)\tilde{A}(s)$, $s \geqslant 0$ for q in Q_δ.

We use the following general notation. By an operator A we mean a linear transformation $A: D_A \to H$ where H is a Hilbert space and D_A, the domain of A, is a dense subspace. A closed means that the graph of A is closed. $\|f\|_{D_A}$ denotes $\|f\| + \|Af\|$ for $f \in D_A$ where $\|\ \|$ denotes the norm of H. $\sigma(A)$ denotes the spectrum of A. If A is bounded, we assume $D_A = H$ and $\|A\|$ denotes the usual operator norm. $A \mid X$ denotes the operator on H with value Af for $f \in X \cap D_A$. For a bounded linear transformation $S: X \to Y$ between two Banach spaces, $\|S\|$ denotes the usual operator norm and $S \mid X_0$ denotes the restriction of S to a subspace X_0 of X. We denote $\int_0^\infty$ by $\int$. If E and F are sets, $E \backslash F$ denotes the elements of E not in F. For a real interval J, $C^{(j)}J$ denotes all complex-valued functions on J such that the function and its derivatives of order $\leqslant j$ are continuous, and $C_c^{(j)}J$ denotes the functions in $C^{(j)}J$ with compact support in J.

Throughout we use

$$\int_a^b (lu)v\,dx - \int_a^b u(lv)\,dx = -(u'v - uv')|_a^b \tag{2.4}$$

for $u, v \in C^{(2)}[a, b]$, $0 \leqslant a < b$.

Finally, we give some properties of $\sigma(T_q)$ in four lemmas.

2.3. LEMMA. *Suppose* $\int |q(t)|\,dt < \infty$. *Then*

$$\sigma_0 \subset \sigma(T) \subset \sigma_0 \cup [0, \infty)$$

where $\sigma_0 = \{s^2 : \operatorname{Im} s > 0, y(s, 0) = 0\}$. *In particular,* σ_0 *is a countable set of isolated points.*

This result is contained in Theorems 3.3.1, 3.4.1 and 3.5.2 in [5]. The proof involves showing that the Green's function is a kernel for a bounded operator which is the inverse of $T - \lambda$.

2.4. LEMMA. *If* $\int |q|\,dx < \infty$, $\lambda \in \sigma_0$ *and* P *is the spectral projection corresponding to* T *and* $\{\lambda\} \subset \sigma(T)$, *then* PL^2 *is finite dimensional.*

The Green's function shows that λ is a pole of the resolvent. Thus, the lemma follows from Theorem 24, which is also true in the unbounded case, on page 576 of [2] and the fact that each eigenspace of T has dimension one.

2.5. LEMMA. *If* $q \in Q_\delta$, *then* σ_0 *is finite.*

This follows from Lemma 5.5 and the assumption that A and $\tilde{A}$ are bounded away from 0 on $0 \leqslant s < \infty$.

2.6. LEMMA. *Suppose* σ_0, σ_{00} *correspond to* q, q_0 *in* Q_δ, *respectively. Then* $\sigma_0 \to \sigma_{00}$ *as* $|q - q_0|_Q \to 0$.

This follows from Lemmas 5.5 and 5.7.

3. THE THEOREM

Throughout this section T denotes the operator corresponding to q and the boundary condition $u(0) = 0$ as in Section 1. In this section we state and prove the one theorem of this paper, which, together with the discussion below, completely describes the structure of T. For basic definitions and notation, see Sections 1 and 2.

DEFINITION. Suppose that A_j is a linear operator on a Hilbert space H_j

with domain D_{A_j}, $j=1, 2$. We say that A_1 is *similar* to A_2 via the similarity transform S if S is a linear transformation with the following properties:

(i) $S: H_1 \to H_2$ is one to one, onto and bicontinuous;

(ii) $S: D_{A_1} \to D_{A_2}$ is one to one, onto and bicontinuous relative to the domain norms;

(iii) $SA_1x = A_2Sx$, $x \in D_{A_1}$.

By Lemma 2.3, $\sigma(T) \subset \sigma_0 \cup [0, \infty)$ where σ_0 is a finite set disjoint from $[0, \infty)$. Let P be the spectral projection corresponding to the operator T and $\sigma_0 \subset \sigma(T)$ (see Introduction). Let $H_0 = PL^2$ and $H_1 = (I-P)L^2$ ($L^2 = L^2(dx)$). By Lemmas 6.1 and 6.2, S and $S^{\check{}}$ are well defined and bounded. Let $S_1 = S \mid H_1$.

Let T_1 denote the operator on H_1 defined by

$$T_1 f = Tf, \qquad f \in D_T \cap H_1.$$

From (iv) of the following paragraph, it follows that T_1 is well defined as an operator on H_1. Note that in particular $D_{T_1} = D_T \cap H_1$. The main theorem, which is stated below, describes the structure of T_1 for certain q.

From Sections 3, 9 in Chapter VII of [2], we have the following conclusions: (i) $H_0, D_T \cap H_1$ are closed in D_T with the domain norm; (ii) $D_T = H_0 \oplus D_T \cap H_1$; (iii) $T: H_0 \to H_0$; (iv) $T_1: D_T \cap H_1 \to H_1$; (v) T_1 is closed and dense as an operator on H_1; (vi) $\sigma(T \mid H_0) = \sigma_0$, $\sigma(T \mid H_1) = \sigma(T) \backslash \sigma_0$. Furthermore, by Lemma 2.4, H_0 is finite dimensional.

The structure of T is described by the previous paragraph together with a description of the structure of T_1, which is the content of the one theorem in this paper. We state this theorem below.

Recall from Section 2 that Q_δ denotes all continuous complex-valued functions q on $[0, \infty)$ such that $\int t \mid q(t) \mid dt < \infty$ and

$$|A(s)|, \qquad |\tilde{A}(s)| \geqslant \delta$$

where $\delta > 0$. A and $\tilde{A}$ are determined by q and the boundary condition $u(0) = 0$, and are defined in Section 2.

THEOREM. *If q is in some Q_δ ($\delta > 0$), then T_1 is similar to M via the similarity transform S_1. Furthermore, $S_1^{-1} = S^{\check{}}$.*

COROLLARY. *If q is in some Q_δ, then $\sigma(T_1) = [0, \infty)$.*

To prove the corollary we note that by the theorem $\sigma(T_1) = \sigma(M)$ and it is easily shown that $\sigma(M) = [0, \infty)$.

The proof of the theorem follows from Lemmas 7.1, 10.1 and 10.2. In the Introduction we gave a brief description of how the lemmas in the paper connect together.

4. An Inequality

The purpose of this section is to establish the inequality

$$\int_0^\infty \int_0^\infty \left(\int_x^\infty |[(e^{i2s(t-x)} - 1)/i2s]\, q(t)|\, dt\right)^2 dx\, ds \leqslant 3 \left(\int_0^\infty t\, |q(t)|\, dt\right)^2. \tag{4.1}$$

This inequality is used in the next section. Clearly,

$$|(e^{isu} - 1)/s| \leqslant \min(2/s, u) \tag{1}$$

for $s > 0$, $u \geqslant 0$, and

$$\int_x^\infty \min(1/s, t)\, |q(t)|\, dt = \int_x^\infty \min(1/st, 1)\, t\, |q(t)|\, dt$$

for $s, t > 0$. Let

$$f_0(x, s) = \int_x^\infty \min(1/st, 1)\, F(t)\, dt$$

for $F(t) \geqslant 0$ and $\int_0^\infty F(t)\, dt < \infty$. It suffices to show

$$\int_0^\infty \int_0^\infty f_0(x, s)^2\, dx\, ds \leqslant 3 \left(\int_0^\infty F(t)\, dt\right)^2. \tag{4.2}$$

Now $f_0 = f_1 + f_2 + f_3$ where

$$f_1(x, s) = \int_x^{1/s} F(t)\, dt, \qquad f_2(x, s) = \int_{1/s}^\infty (1/st)\, F(t)\, dt$$

for $x \leqslant 1/s$ and $f_1, f_2 = 0$ otherwise; and

$$f_3(x, s) = \int_x^\infty (1/st)\, F(t)\, dt$$

for $x > 1/s, f_3 = 0$ otherwise. We now examine f_1, f_2 and f_3. We need

$$\int_0^\infty t \left| \int_t^\infty (1/u)\, F(u)\, du \right|^2 dt \leqslant \frac{1}{2} \left(\int_0^\infty F(t)\, dt \right)^2. \tag{4.3}$$

To prove (4.3), suppose $g(t) \geqslant 0$ and

$$\int_0^\infty t g(t)^2\, dt = 1.$$

Then

$$\int_0^\infty t \left(\int_t^\infty (1/u)\, F(u)\, du \right) g(t)\, dt \leqslant (1/\sqrt{2}) \int_0^\infty F(t)\, dt$$

since

$$\int_0^u t g(t)\, dt \leqslant (1/\sqrt{2})\, u$$

and (4, 3) now follows.

By applying (4.3) we get

(2) $$\int_0^\infty \int_{1/s}^\infty f_3(x, s)^2\, dx\, ds \leqslant \frac{1}{2} \left(\int_0^\infty F(t)\, dt \right)^{1/2}.$$

Now

$$\int_0^\infty \int_0^{1/s} f_2(x, s)^2\, dx\, ds \leqslant \int_0^\infty s^{-3} \left(\int_{1/s}^\infty (1/t)\, F(t)\, dt \right)^2 ds$$

and by changing variables $(v = 1/s)$ and applying (4.3) we get

(3) $$\int_0^\infty \int_{1/s}^\infty f_2(x, s)^2\, dx\, ds \leqslant \frac{1}{2} \left(\int_0^\infty F(t)\, dt \right)^2.$$

Suppose $G(x, s) \geqslant 0$ and $\int G(x, s)^2\, dx\, ds = 1$. Then

$$\int_0^{1/t} \int_0^t G(x, s)\, dx\, ds \leqslant 1.$$

Thus,

$$\int_0^\infty \int_0^{1/s} f_1(x, s)\, G(x, s)\, dx\, ds \leqslant \int_0^\infty F(t)\, dt.$$

Thus,

$$\int_0^\infty \int_0^{1/s} f_1(x,s)^2\,dx\,ds \leqslant \left(\int_0^\infty F(t)\,dt\right)^2. \tag{4}$$

Finally, (4.2) follows from (2), (3) and (4).

5. Properties of v

The purpose of this section is to obtain several properties of the function v which will be used throughout the paper. Lemmas 5.1–5.4 are preliminary, Lemmas 5.5–5.9 are properties of v assuming that $q \in Q$ and Lemmas 5.10 and 5.11 are properties of v assuming the stronger condition (5.2). See Sections 1 and 2 for basic notation.

Let C_∞ denote the space of complex valued continuous functions on $[0, \infty)$ which have a limit as $x \to \infty$. We give C_∞ the uniform norm. For $q \in Q$ and $\operatorname{Im} s \geqslant 0$ we define the operator $V \equiv V_{sq}$ on C_∞ by

$$\begin{aligned} Vv(x) &= \int_x^\infty ((e^{i2s(t-x)} - 1)/i2s)\, q(t)\, v(t)\,dt, \qquad s \neq 0, \\ &= \int_x^\infty (t-x)\, q(t)\, v(t)\,dt, \qquad s = 0. \end{aligned}$$

If A is a bounded operator on C_∞, we use $\|A\|$ below to denote the usual operator norm.

5.1. Lemma. *For* $\operatorname{Im} s \geqslant 0$, $q \in Q$,

(i) $\|V_{sq}\| \leqslant \int t\,|q(t)|\,dt$,

(ii) $\|V_{sq}\| \leqslant (1/|s|) \int |q(t)|\,dt$, $s \neq 0$.

5.2. Lemma. *The map* $(s, q) \to V_{sq}$ *is continuous from* $\{s: \operatorname{Im} s \geqslant 0\} \times Q$ *into the bounded operators on* C_∞ *(the topology of* Q *is induced by the norm* $\int (1+t)\,|q(t)|\,dt$*).*

The proofs of Lemmas 5.1 and 5.2 are routine.

5.3. Lemma. *For* $\operatorname{Im} s \geqslant 0$ *and* $q \in Q$,

$$\sigma(V_{sq}) = \{0\}.$$

Clearly V_{sq} has the form

$$Vu(x) = \int_x^\infty K(x, t)\, h(t)\, u(t)\, dt$$

where K is bounded and continuous and $h \in L^1[0, \infty)$. Thus, V is a compact operator since the image of the unit ball is an equicontinuous family. A computation shows that V has no nonzero eigenvalues. Thus, $\sigma(V) = \{0\}$ by page 579 of [2].

5.4. LEMMA. *Suppose that Q_0 is a compact subset of Q. Then $\|(I - V_{sq})^{-1}\|$ is bounded for* $\operatorname{Im} s \geqslant 0$, $q \in Q_0$.

Since $V_{sq}\| < \frac{1}{2}$ for $|s|$ sufficiently large, this lemma follows from Lemma 5.3, the fact that the operator inverse is continuous and the use of compactness.

Let $q \in Q$ and $\operatorname{Im} s \geqslant 0$. By Lemma 5.3, the integral equation

$$v(s, x) = 1 + \int_x^\infty [(e^{i2s(t-x)} - 1)/i2s]\, q(t)\, v(s, t)\, dt \tag{5.1}$$

(the integrand is interpreted as $(t - x)\, q(t)\, v(0, t)$ when $s = 0$) has a unique solution which we denote by $v(s, x)$. Note that this is the same integral equation as (2.1). Throughout the paper we use the properties of v given in the following seven lemmas.

5.5. LEMMA. *Suppose $q \in Q$. For* $\operatorname{Im} s \geqslant 0$, $x \geqslant 0$,

(i) *$v(s, x)$ is bounded and continuous,*

(ii) $v'(s, x) = -\int_x^\infty e^{i2s(t-x)} q(t)\, v(s, t)\, dt$,

(iii) $v''(s, x) = q(x)\, v(s, x) + \int_x^\infty (i2s)\, e^{i2s(t-x)} q(t)\, v(s, t)\, dt \quad (' = d/dx)$.

In particular, v' and v'' are continuous.

We get (i) from Lemma 5.4 and (ii) and (iii) by differentiating (5.1).

5.6. LEMMA. *If $q \in Q$, then $\iint |v(s, x) - 1|^2\, ds\, dx < \infty$.*

This follows from Lemma 5.5(i) and (4.1) of Section 4.

5.7. LEMMA. *Let v, v_0 correspond to $q, q_0 \in Q$. Then*

(i) *there exist M, $\delta > 0$ such that for $|q - q_0|_Q < \delta$, $|v(s, x)| \leqslant M$,* $\operatorname{Im} s \geqslant 0$, $x \geqslant 0$;

(ii) $\sup |v(s, x) - v_0(s, x)| \to 0$ *as* $|q - q_0|_Q \to 0$ *where* $\sup$ *is over* $\operatorname{Im} s \geqslant 0$, $x \geqslant 0$.

If (i) were false, then for some s_n and q_n with $|q_n - q_0|_Q \to 0$, the sequence $v_n(s_n, \cdot) = (I - V_{s_n q_n})^{-1}1$ would be unbounded in C_∞; however, this contradicts Lemma 5.4. To prove (ii) we note that

$$v - v_0 = (I - V_{sq_0})^{-1}((V_{sq} - V_{sq_0})v).$$

Thus, (ii) follows from Lemmas 5.4 and 5.1 and (i) of Lemma 5.7.

5.8. LEMMA. *Suppose* $q_0 \in Q_\delta$, $0 < \delta_1 < \delta$ *and* $q \in Q$. *Then* $q \in Q_{\delta_1}$ *if* $|q - q_0|_Q$ *is sufficiently small.*

This follows from Lemma 5.7, the fact that $A(s) = v(s, 0)$ and a similar argument for $\tilde{A}$.

5.9. LEMMA. *Suppose* $q \in Q$, *that* $v(s,x)$ *solves* (5.1) *for* $\operatorname{Im} s \geqslant 0$ *and* $\tilde{v}(s,x)$ *solves* (2.3) *for* $\operatorname{Im} s \leqslant 0$. *Then* $v(0,x) = \tilde{v}(0,x)$; *in particular,* $A(0) = \tilde{A}(0)$.

This follows from the fact that (5.1) and (2.3) become the same equation when $s = 0$.

For the last two lemmas we will assume that q satisfies

$$|q(t)| \leqslant Ce^{-\varepsilon_1 t}, \qquad t \geqslant 0. \tag{5.2}$$

5.10. LEMMA. *Suppose that* q *satisfies* (5.2) *and* $|\operatorname{Im} 2s| < \varepsilon_2$ $(\varepsilon_2 < \varepsilon_1)$. *Then*

(i) (5.1) *has a unique solution* v *such that* $s \to v(s, \cdot)$ *is a bounded analytic function from* $|\operatorname{Im} 2s| < \varepsilon_2$ *into* C_∞;

(ii) $v(s,x) \to 1$ *as* $|s| \to \infty$ *uniformly for* $x \geqslant 0$;

(iii) *the solutions of* (5.1) *and* (2.2) *agree for* $-\varepsilon_2 < \operatorname{Im} 2s < 0$, $x \geqslant 0$;

(iv) $A(s)$, $\tilde{A}(s)$ *are bounded on* $|\operatorname{Im} 2s| < \varepsilon_2$.

The proof of (i) is similar to the proof of Lemma 5.4. We get (ii) from (i) and (5.1). We now prove (iii). Suppose that $-\varepsilon_2 < \operatorname{Im} 2s < 0$ and that $v(s, \cdot)$ and $v_1(s, \cdot)$ solve (5.1) and (2.2), respectively. Let $y(s,x) = e^{isx}v(s,x)$ and $y_1(s,x) = e^{isx}v_1(s,x)$. Since y_1 solves the equation $-u'' + qu = s^2 u$, $y_1 = ay + b\tilde{y}$ (see Section 2); since y and $y_1 \sim e^{isx}$ as $x \to \infty$, clearly $y_1 = y$. It follows from (i) and (iii) ($A(s) = y(s, 0)$) that A is bounded and a similar argument applies to $\tilde{A}$.

5.11. LEMMA. *Suppose that* q *satisfies* (5.2). *Then* $w(s,x)$ *is bounded for* $|\operatorname{Im} 2s| < \varepsilon_2 < \varepsilon_1$ *and for* x *in any bounded interval* $(x \geqslant 0)$.

This follows from (i) of Lemma 5.10 and the definition of w.

6. Bounds for S, $S^{\vee}$

The purpose of this section is Lemmas 6.1 and 6.2. Recall that S and $S^{\vee}$ are defined in Section 1 and that definitions related to w are given in Section 2.

6.1. Lemma. *If $q \in Q$, then S is a well-defined bounded linear transformation from $L^2(dx)$ into $L^2(ds)$.*

6.2. Lemma. *If $q \in Q_\delta$ (some $\delta > 0$), then $S^{\vee}$ is a well-defined bounded linear transformation from $L^2(ds)$ into $L^2(dx)$.*

We now prove Lemma 6.1. Let $f \in C_c[0, \infty)$. By definition,

$$Sf(s) = \int w(s, x) f(x)\, dx.$$

We can express w as

$$w(s, x) = \tilde{A}(s)\, e^{isx} v(s, x) - A(s)\, e^{-isx} \tilde{v}(s, x).$$

Consider

$$\begin{aligned} g(s) &\equiv \tilde{A}(s) \int e^{isx} v(s, x) f(x)\, dx \\ &\equiv \tilde{A}(s) \int e^{isx} (v(s, x) - 1) f(x)\, dx + \tilde{A}(s) \int e^{isx} f(x)\, dx. \end{aligned}$$

From Lemma 5.5 we know that $\tilde{A}$ is bounded; this, Lemma 5.6 and the Plancherel theorem shows that $\| g \| \leqslant C \| f \|$ where the norms are the L^2 norm. The other term in w can be treated in the same manner.

For the second lemma, recall that

$$S^{\vee} g(x) = \int w(s, x)\, g(s)\, W(s)\, ds$$

where $W(s) = -1/2\pi A(s) \tilde{A}(s)$. By assumption, $|A|$ and $|\tilde{A}|$ are bounded away from 0 for $0 < s < \infty$; thus, W is bounded. The proof now proceeds like the first proof.

7. $ST = MS$

In this section we prove Lemmas 7.1 and 7.2 below. See Sections 1 and 2 for basic definitions. We have introduced S, $S^{\smile}$ in Section 1 and have shown that they are well defined and bounded in Section 6.

7.1. LEMMA. *Suppose that $q \in Q$. If $f \in D_T$, then*

(i) $Sf \in D_M$,

(ii) $STf = MSf$,

(iii) $\|Sf\|_{D_M} \leqslant \|S\| \, \|f\|_{D_T}$.

7.2. LEMMA. *Suppose that $q \in Q_\delta$ for some $\delta > 0$ and q is bounded. If $g \in D_M$, then*

(i) $S^{\smile} g \in D_T$,

(ii) $S^{\smile} Mg = TS^{\smile} g$,

(iii) $\|S^{\smile} g\|_{D_T} \leqslant \|S^{\smile}\| \, \|g\|_{D_M}$.

To prove Lemma 7.1, let $f \in C_c^{(2)}[0, \infty)$ with $f(0) = 0$. In particular, $f \in D_T$. By (2.4)

$$STf(s) = \int_0^\infty w(s, x) \, lf(x) \, dx = \int_0^\infty lw(s, x) f(x) \, dx$$

$$= s^2 \int_0^\infty w(s, x) f(x) \, dx;$$

this, the fact that S is bounded and a closure argument prove (i) and (ii). The proof of (iii) follows easily from (ii) and the definition of the domain norm of an operator.

The proof of Lemma 7.2 is based on the following two lemmas.

7.3. LEMMA. *If $g \in C_c[0, \infty)$, then $f(x) \equiv \int_0^\infty w(s, x) \, g(s) \, ds$ is in $C^{(2)}[0, \infty)$ and $f''(x) = \int_0^\infty w''(s, x) \, g(s) \, ds$ ($''$ denotes d^2/dx^2).*

7.4. LEMMA. *Suppose q is bounded (and continuous). If $f \in C^{(2)}[0, \infty)$, $f(0) = 0$ and $f, lf \in L^2[0, \infty)$, then $f \in D_T$.*

We prove the last two lemmas after we prove Lemma 7.2. Suppose $g \in C_c[0, \infty)$ and let

$$f(x) = S^{\smile} g(x) = \int_0^\infty w(s, x) \, g(s) \, W(s) \, ds.$$

Thus, $f(0)=0$ since $w(0,s)=0$, $f \in L^2$ since $S^{\check{}}$ is bounded and $f \in C^{(2)}[0,\infty)$ by Lemma 7.3. Also, by Lemma 7.3,

$$(1) \qquad lf(x)=\int_0^\infty lw(s,x)\,g(s)\,W(s)\,ds = S^{\check{}}s^2 g;$$

thus, $lf \in L^2[0,\infty)$. Thus, $lf \in D_T$ by Lemma 7.4. Also from (1) we have $TS^{\check{}}g = S^{\check{}}Mg$. A routine closure argument completes the proof of (i) and (ii). The proof of (iii) uses (ii) and the definition of the domain norm.

From the definition of w and Lemma 5.5, we see that $w'(s,x)$ and $w''(s,x)$ are continuous; so Lemma 7.3 follows since g has compact support.

To prove Lemma 7.4 we need

7.5. LEMMA. *If $u \in C^{(2)}[0,\infty)$ and $u, u'' \in L^2[0,\infty)$, then there is a sequence $t_1, t_2, \ldots$ with $t_n \to \infty$ such that $|u(t_n)|+|u'(t_n)|+|u''(t_n)| \to 0$ as $n \to \infty$.*

To prove this lemma, it suffices to show that $u'(t) \to 0$ as $t \to \infty$. Let $a \geqslant 0$ and suppose $|u'(x_0)| \geqslant |u'(x)|$ where x_0, $x \in [a, a+1]$. In particular,

$$(2) \qquad |u'(x)-u'(x_0)| \leqslant \left(\int_a^{a+1} |u''|\,dx\right)^{1/2}.$$

Either

$$(3) \qquad |u'(x_0)| \leqslant 2\left(\int_a^{a+1} |u''|\,dx\right)^{1/2}$$

or (3) fails. Because of (2), if (3) fails, then

$$(4) \qquad \frac{1}{2}|u'(x_0)| \leqslant \left|\int_a^{a+1} u'v\,dx\right|$$

where $v \geqslant 0$ and $\int_a^{a+1} v\,dx = 1$. Also,

$$(5) \qquad \int_x^\infty u''v - uv'' = -u'(x)\,v(x)+u(x)\,v'(x)$$

where $v \in C_c^{(2)}[0,\infty)$. If $v \in C^{(2)}[0,\infty)$, $v \geqslant 0$, $v=0$ off $[a,a+1]$ and $\int v\,dx = 1$, then by (5) we have

$$(6) \qquad \int_a^{a+1} u'v\,dx = \int_a^{a+1} h(x)+u(x)\,v(x)\,dx$$

where $h(x) = -\int_x^\infty (u''v-uv'')\,dx$. Thus, either (3) holds or (4) and (6) hold and we conclude that $u(t) \to 0$ as $t \to \infty$. So, Lemma 7.5 is proved.

Using the fact that q is bounded, it is easy to see that $f'' \in L^2$. Thus, Lemma 7.5 applies. Let t_n be as in Lemma 7.5. Define v_n as follows: $v_n = f$ for $x \leqslant t_n$, $v_n = 0$ for $x > t_n + 1$ and $v_n = p$ for $t_n < x \leqslant t_n + 1$ where p is the polynomial of degree 5 such that $p^{(j)}(t_n) = f^{(j)}(t_n)$ and $p^{(j)}(t_n + 1) = 0$ for $j = 0, 1, 2$. It is now easy to see that $v_n(0) = 0$, $v_n \in C_c^{(2)}$, $\|f - v_n\|_{L^2}$ and $\|f'' - v_n''\|_{L^2} \to 0$ as $n \to \infty$. Thus, Lemma 7.4 is proved.

8. $S^{\check{}} S = I - P$ FOR SPECIAL q

Throughout this section we assume that q is continuous and that for some $C, \delta, \varepsilon_1 > 0$,

$$|q(t)| \leqslant Ce^{-\varepsilon_1 t}, \qquad 0 \leqslant t < \infty, \tag{8.1}$$

$$|A(s)|, |\tilde{A}(s)| \geqslant \delta, \qquad 0 \leqslant s < \infty. \tag{8.2}$$

Let T correspond to q as in Section 1. Recall that $\sigma_0 = \sigma(T) \backslash [0, \infty)$, that by Lemma 2.5 σ_0 is finite and that P is the spectral projection corresponding to the operator T and $\sigma_0 \subset \sigma(T)$ as in Section 1. The purpose of this section is the proof of

8.1. LEMMA. *If q is continuous and satisfies* (8.1) *and* (8.2), *then* $S^{\check{}} S = I - P$.

For each $b > 0$, let T_b be the operator on $L^2[0, b]$ obtained by taking the closure of l restricted to $\{u: u \in C^{(2)}[0, b],\ u(0) = 0 = u(b)\}$. Choose ε_2, $0 < \varepsilon_2 < \varepsilon_1$, such that for $\Gamma \equiv \{s: \operatorname{Re} s \geqslant 0, |\operatorname{Im} 2s| \leqslant \varepsilon_2\}$, the set $\{s^2: s \in \Gamma\}$ is disjoint from σ_0 and

$$|A(s)|, |\tilde{A}(s)| \geqslant \delta, \qquad s \in \Gamma, \tag{8.3}$$

for some $\delta > 0$. To get (8.3) we use (8.2) and Lemma 5.10.

For b sufficiently large, let P_b be the spectral projection (see reference in Introduction) corresponding to the operator T_b and $\sigma(T_b) \backslash \{s^2: s \in \Gamma\}$, which is finite by Lemma 8.5 below. To prove the lemma, it suffices to show that

$$(f, h) = (Pf, h) + (S^{\check{}} Sf, h)$$

for $f \in C_c^{(2)}(0, \infty)$ and $h \in C_c[0, \infty)$. However, since $f = P_b f + (I - P_b) f$ for $f \in L^2[0, b]$, it suffices to show that

$$(P_b f, h)_b \to (Pf, h) \qquad \text{as} \quad b \to \infty, \tag{8.4}$$

$$((I - P_b) f, h)_b \to (S^{\check{}} Sf, h) \qquad \text{as} \quad b \to \infty, \tag{8.5}$$

with f and h as above ($(f, g)_b$ denotes $\int_0^b f(x) \overline{g(x)}\, dx$).

To prove (8.4) and (8.5) we need the following four lemmas.

8.2. LEMMA. *There exist $b_0 > 0$ positive integers j_0 and k_0 and complex numbers s_{bj} for $j = j_0 + 1, j_0 + 2,...$ and $b \geqslant b_0$ such that:*

(i) *$s_{bj}, j > j_0$, are distinct zeros of $w(\cdot, b)$ in $\Gamma \backslash \{0\}$;*

(ii) *$s_{bj} = (j\pi + m(b, j))/b$ where*

(iii) *$m(b, j)$ is bounded for $j > j_0$, $b \geqslant b_0$ and*

(iv) *$m(b, j) \to 0$ as $b \to \infty$ for each j;*

(v) *Z_b, the set of zeros of $w(\cdot, b)$ in $\Gamma \backslash \{0\}$ which are not listed in* (i), *has no more than k_0 elements;*

(vi) *$s = (j\pi + m(b, s))/b$ for s in Z_b where $b \geqslant b_0$, $1 \leqslant j \leqslant j_0$, $m(b, s) \to 0$ as $b \to \infty$ uniformly for $s \in Z_b$.*

Proof. First extend the set Γ to a set $\Gamma_1 \equiv \{s: -\delta_1 \leqslant \operatorname{Re} s, |\operatorname{Im} 2s| \leqslant \varepsilon_2\}$ for $\delta_1 > 0$. We may choose δ_1 so that $A, \tilde{A}$ are bounded away from zero on Γ_1. Then for b large, $F(s) \equiv (1/2i) \log(A\tilde{v}/\tilde{A}v)$ is analytic on Γ_1 by Lemma 5.10. From the definition of w we see that s is a zero of $w(\cdot, b)$ in Γ if and only if

$$(1) \qquad s = (j\pi + F(s))/b, \qquad \text{some integer } j.$$

Suppose $|F| \leqslant M$ on Γ_1 and $r > M$. Assume that the distance from $[0, \infty)$ to the boundary of $\Gamma_1 > r/b_0$. Let $D(z_0, a) \equiv \{z: |z - z_0| < a\}$ in the complex plane. Suppose $j_0 \pi \geqslant r$. From Rouche's theorem, Eq. (1) has exactly one root in $D(j\pi/b, r/b) \subset \Gamma$ for $j > j_0$; let s_{bj} denote this root for b sufficiently large. Now suppose that s is a zero of $w(\cdot, b)$ in Γ. Then s is a root of (1) for some j where $j \geqslant -r/\pi$; hence, $s \in D(j\pi/b, r/b)$, which we may assume is contained in Γ_1. If $j > j_0$, then s is already listed in (i). Again by Rouche's theorem, (1) has exactly one root in $D(j\pi/b, r/b)$. The above remarks establish the lemma except for (iv) and (vi). From the above and Lemmas 5.9 and 5.10 we get (iv) and (vi) except for the conclusion that $j \geqslant 1$ in (vi), which follows if we show:

$$(*) \qquad \text{For some } b_0,\ \varepsilon > 0,\ w(s, b) \neq 0 \text{ for } b \geqslant b_0,\ 0 < |s| < \varepsilon/b.$$

From the definition of w we have

$$w(s, b)\, e^{isb}/\tilde{A}(s)\, v(s, b) = e^{i2sb} - A(s)\, \tilde{v}(s, b)/\tilde{A}(s)\, v(s, b);$$

and, by Lemma 5.9 and a Taylor expansion, the right-hand side of this latter equation $\neq 0$ for some b_0, ε in $(*)$. Thus $(*)$ is proved.

8.3. LEMMA. *If q is a continuous complex-valued function on $[0, \infty)$ and*

$\int_0^\infty t^2 |q(t)|\, dt < \infty$, then each nonzero solution of $u'' = qu$ has a finite number of zeros in $[0, \infty)$.

This can be proved by carefully examining the corresponding integral equations

$$u(x) = a + bx - \int_x^\infty (x-t)\, q(t)\, u(t)\, dt$$

where a, b are complex numbers.

8.4. Lemma. *Let s_{bj} and Z_b be as in Lemma 8.2. Then $(1/b)\int_0^b w(s,x)^2\, dx + 2A(s)\tilde{A}(s) \to 0$ as $b \to \infty$ uniformly for $s = s_{bj}$, $j = 1, 2, \ldots$ or $s \in Z_b$.*

Proof. First let $s = s_{bj}$. Since $w = \tilde{A}e^{isx}v - Ae^{-isx}\tilde{v}$ and v, $\tilde{v} \to 1$ as $x \to \infty$ uniformly for s in Γ, the problem reduces to showing that

$$(1/b)\int_0^b e^{2i(s_{bk})x}\, dx \to 0 \qquad \text{as} \quad b \to \infty \tag{2}$$

uniformly in k. From Lemma 8.2 we conclude that

$$(1/b)\int_0^b e^{2i(s_{bk})x}\, dx = \int_0^1 e^{i2k\pi x} e^{i2m(b,k)x}\, dx. \tag{3}$$

For the right-hand side of (3), we see that it is small for large k since $m(b, k)$ is bounded and it tends to 0 as $b \to \infty$ for each fixed k since $m(b, k) \to 0$ as $b \to \infty$ by Lemma 8.2. Because of (vi) in Lemma 8.2, the case of $s \in Z_b$ can be treated in the same manner.

Let γ denote a simple closed curve in the complex plane which lies outside $\{^2 : s \in \Gamma\}$ and which contains $\sigma_0 \equiv \sigma(T)\backslash[0, \infty)$ in its interior. We denote the interior of γ by int γ.

8.5. Lemma. *For b sufficiently large:*

(i) *$\sigma(T_b) = \{s^2 : s \neq 0, w(s, b) = 0\}$ and each point in $\sigma(T_b)$ is an eigenvalue;*

(ii) *$\sigma(T_b) \subset \operatorname{int}\gamma \cup \{s^2 : s \neq 0, s \in \Gamma\}$;*

(iii) *$\sigma(T_b) \cap \operatorname{int}\gamma$ is finite;*

(iv) *the generalized eigenspace of T_b corresponding to each point in $\sigma(T_b) \cap \{s^2 : s \in \Gamma\}$ has dimension $=1$.*

Proof. It follows from Lemma 8.3 that 0 is not an eigenvalue of T_b; thus, (i) follows from properties of the Green's function. One can prove (ii) by using y, $\tilde{y}$ to express the solutions of $-u''+qu=\lambda u$ with $u(0)=0$ and $v(b)=0$ or $u\in L^2[0,\infty)$. (iii) follows from (i) and the analyticity of w in the s variable. To prove (iv), we first show that if

$$\int_0^b w(s,x)^2\,dx \neq 0, \tag{4}$$

and s^2 is an eigenvalue of T_b, then the generalized eigenspace of s^2 has dimension $=1$. Because the eigenspace has dimension $=1$ (T_b is a differential operator of order two), if the generalized eigenspace had dimension >1, then there would be functions u, v such that both have value $=0$ at 0 and b and $(l-\lambda)u=0$ and $(l-\lambda)v=u$ $(\lambda=s^2)$. Thus,

$$\begin{aligned}\int_0^b u^2\,dx &= \int_0^b [(l-\lambda)v]u\,dx\\ &= \int_0^b v[(l-\lambda)u]\,dx-(v'u-vu')\Big|_0^b=0,\end{aligned}$$

and the assertion is proved.

To prove (iv), it now suffices to show that if b is sufficiently large and $s^2\in\sigma(T_b)$ with $s\in\Gamma$, then (4) holds; but this follows from Lemma 8.4 and (8.3). So, Lemma 8.5 is proved.

We now prove (8.4). Give the curve γ, which is defined above, the usual orientation of complex variable theory. Then

$$P=(1/2\pi i)\int_\gamma (\lambda-T)^{-1}\,d\lambda.$$

By Lemma 8.5, for b sufficiently large,

$$\sigma(T_b)\subset \operatorname{int}\gamma\cup\{s^2: s\in\Gamma\};$$

thus,

$$P_b=(1/2\pi i)\int_\gamma (\lambda-T_b)^{-1}\,d\lambda$$

for b sufficiently large. Let $G(\lambda,x,t)$ and $G_b(\lambda,x,t)$ denote the Green's functions corresponding to $(T-\lambda)^{-1}$ and $(T_b-\lambda)^{-1}$, respectively. Thus, to prove (8.4), it suffices to show that

$$G(\lambda,x,t)-G_b(\lambda,x,t)\to 0 \qquad \text{as} \quad b\to\infty \tag{5}$$

uniformly for $\lambda\in\gamma$ and x,t bounded. This is easily proved by directly comparing the Green's functions for λ in γ in the cases ($\lambda = s^2$) Im $s > 0$ and Im $s < 0$. So, (8.4) is proved.

The rest of this section is devoted to the proof of (8.5). Let s_{bj} and Z_b be as in Lemma 8.2. From Lemma 8.4 and (8.3) we have

$$\int_0^b w(s,x)^2\,dx \neq 0$$

for $s = s_{bj}$ or $s \in Z_b$ and b sufficiently large. Furthermore, by Lemma 8.3, 0 is not an eigenvalue of T_b for b large. By Lemma 8.5, each generalized eigenspace of T_b corresponding to one of the eigenvalues in $\{s^2 : s \in \Gamma\}$ has dimension $=1$. The projections onto these one-dimensional generalized eigenspaces are simultaneously similar to orthogonal projections by Corollary 1 on page 442 of [6] and Lemma 2 on page 1947 of [3]. Let

$$W_{bj} = 1\Big/\int_0^b w(s_{bj},x)^2\,dx$$

and

$$W_{bs} = 1\Big/\int_0^b w(s,x)^2\,dx, \qquad s \in Z_b.$$

Thus,

$$(I-P)f = \sum_{j=1}^{\infty} w(s_{bj},x)\left(\int_0^b w(s_{bj},x')\,f(x')\,dx'\right) W_{bj} + \sum_0$$

where $\sum_0$ denotes the first sum with s_{bj} replaced by $s \in Z_b$ and W_{bj} replaced by W_{bs} and the sum is over $s \in Z_b$ and where the infinite series converges in the space $L^2[0,b]$. From Lemma 8.4, (v) of Lemma 8.2 and the fact that $w(s,x)$ is bounded for x bounded and $s \in Z_b$, we see that $\sum_0 \to 0$ as $b \to \infty$ uniformly for x in bound intervals. Thus, letting $g(s) = \int w(x,s)\,f(x)\,dx$, s complex, to prove (8.5) it suffices to show that

$$\sum_{j=1}^{\infty} w(s_{bj},x)\,g(s_{bj})\,W_{bj} \to \int_0^{\infty} w(s,x)\,g(s)\,W(s)\,ds \tag{8.6}$$

uniformly for x in bounded subsets of $[0,\infty)$. Let Δ_{bj} denote the interval $(j-1)\pi/b < x \leqslant j\pi/b$, $j = 1, 2, \ldots$, $b > 0$. To prove (8.6) it clearly suffices to prove the following hold as $b \to \infty$ uniformly for x in bounded intervals (the sums are over $j = 1, 2, \ldots$):

$$\sum w(s_{bj},x)\,g(s_{bj})\left(W_{bj} - \int_{\Delta_{bj}} W(s)\,ds\right) \to 0; \tag{8.7}$$

$$\sum w(s_{bj}, x) \int_{\Delta_{bj}} (g(s_{bj}) - g(s))\, W(s)\, ds \to 0; \tag{8.8}$$

$$\sum \int_{\Delta_{bj}} (w(s_{bj}, x) - w(s, x))\, g(s)\, W(s)\, ds \to 0. \tag{8.9}$$

The rest of this section is devoted to proving (8.7)–(8.9). The proof of (8.7) is long. For convenience, we state some preliminary facts, (6)–(11).

(6) $\quad w(s, x)$ is bounded for x in bounded intervals and $s \in \Gamma$.

This follows from Lemma 5.10 and the definition of w.

(7) Given $M > 0$, there corresponds $C > 0$ such that $|s_{bj}| \geqslant C(j/b)$ for $j/b \geqslant M$ and b large.

This follows from Lemma 8.2.

(8) $\left| (1/b) \int_0^b w(s_{bj}, x)^2\, dx \right| \geqslant \delta$, $j = 1, 2,\ldots$ for some $\delta > 0$ and b sufficiently large.

This follows from Lemma 8.3 and (8.3).

(9) $\quad |v(s, x) - 1| \to 0 \quad$ as $x \to \infty$ uniformly for $s \in \Gamma$.

This follows from (2.1) and Lemma 5.10.

(10) $\quad (1/b) \sum_{j \leqslant bM} \sup_{s \in \Delta_{bj}} |A(s_{bj}) - A(s)|^2 \to 0$ as $b \to \infty$ for each $M > 0$.

(11) $\quad \sup_{s \in \Delta_{bj}} |s_{bj} - s| \to 0$ as $b \to \infty$ uniformly for $j = 1, 2,\ldots$.

Both (10) and (11) follow from Lemma 8.2.

We now consider (8.7)–(8.9). Because of (6), to prove (8.7) it suffices to show

(12) $\quad (1/b) \sum |g(s_{bj})|^2 \quad$ is bounded as $b \to \infty$,

and

(13) $\quad b \sum \left| W_{bj} - \int_{\Delta_{bj}} W(s)\, ds \right|^2 \to 0 \quad$ as $b \to \infty$

where each sum is over $j = 1, 2,\ldots$. We prove (12) by summing over $j \leqslant bM$ and $j > bM$ separately and applying (6) to the first sum and (6), (7) and (2.4) to the second sum.

We now prove (13). Because of (8) and the fact that A, $\tilde{A}$ are bounded away from 0 on $[0, \infty)$, it suffices to prove

$$(14) \quad (1/b)\sum \sup_{s\in\Delta_{bj}} \left| 2A(s)\tilde{A}(s) + (1/b)\int_0^b w(s_{bj}, x)^2\, dx \right|^2 \to 0 \qquad \text{as } b\to\infty,$$

where $\sum$ is over $j = 1, 2, \ldots$. Now

$$2A(s)\tilde{A}(s) - (1/b)\int_0^b w(s_{bj}, x)^2\, dx$$
$$= 2(A(s)\tilde{A}(s) - A(s_{bj})\tilde{A}(s_{bj})) + 2(1-B)A(s_{bj})\tilde{A}(s_{bj}) + B_1$$

where

$$B = (1/b)\int_0^b v(s_{bj}, x)\,\tilde{v}(s_{bj}, x)\, dx$$

$$B_1 = (1/b)\int_0^b [(\tilde{A}(s_{bj})\,v(s_{bj}, x))^2 e^{i2xs_{bj}} + (A(s_{bj})\,\tilde{v}(s_{bj}, x))^2 e^{-i2xs_{bj}}]\, dx.$$

Thus, to prove (14), it suffices to show that

$$(15) \qquad (1/b)\sum \sup_{s\in\Delta_{bj}} |B_2|^2 \to 0 \qquad \text{as} \quad b\to\infty$$

for the following cases: (1) $B_2 = B - 1$; (2) $B_2 = A(s_{bj})\tilde{A}(s_{bj}) - A(s)\tilde{A}(s)$; and (3) $B_2 = B_1$. For case (1), it suffices to show

$$(16) \qquad (1/b)\sum \left| (1/b)\int_0^b (v(s_{bj}, x) - 1)\, dx \right|^2 \to 0 \qquad \text{as} \quad b\to\infty$$

($\sum$ over $j = 1, 2, \ldots$) and similarly for $\tilde{v}$. To prove (16) we sum over $j \leqslant bM$ and $j > bM$ separately and apply Lemma 5.10 and (9) to the first sum and (2.4) and (7) to the second sum. Case (2) reduces to showing

$$(17) \qquad (1/b)\sum \sup_{s\in\Delta_{bj}} |A(s_{bj}) - A(s)|^2 \to 0 \qquad \text{as} \quad b\to\infty,$$

and similarly for $\tilde{A}$ since A, $\tilde{A}$ are bounded on Γ. For (17), we sum separately over $j \leqslant bM$ and $j > bM$ and apply (10) to the first sum and (2.1) to the second. Since A, $\tilde{A}$ are bounded on Γ and $e^{i2xs_{bj}}$ is bounded, case (3) reduces to (16) and to showing

$$(18) \qquad (1/b)\sum \left| (1/b)\int_0^b e^{i2xs_{bj}}\, dx \right|^2 \to 0 \qquad \text{as} \quad b\to\infty$$

where $\sum$ is over $j = 1, 2, \ldots$. This is proved by summing over $j \geqslant b\varepsilon$ and $j \geqslant b\varepsilon$ and applying Lemma 8.2 and (7). This completes the proof of (8.7).

Since $w(s_{bj}, x)$ is bounded for x in bounded intervals and W is bounded, to prove (8.8) it suffices to show

$$(1/b) \sum \sup_{s \in \Delta_{bj}} |g(s_{bj}) - g(s)| \to 0 \qquad \text{as} \quad b \to \infty \tag{19}$$

where $\sum$ is over $j = 1, 2, \ldots$. To prove (19) we split the sum into three sums corresponding to $j \leqslant b\varepsilon$, $\varepsilon b < j \leqslant bM$ and $j > bM$. For the first sum, we use the fact that g is bounded on Γ by Lemma 5.10 and the definition of w. For the second sum, we use the continuity of g and (11). For the third sum, we use (7) and the fact that

$$g(s) = s^{-2} \int w(s, x)\, lf\, dx \tag{20}$$

by (2.4).

Finally, we prove (8.9). We write the sum as two sums corresponding to $j \leqslant bM$ and $j > bM$. For the second sum we use the boundedness of w and W and (20) and choose M sufficiently large. For the first sum we use the continuity of w and (11).

9. $S^{\check{}} S = I - P$

The purpose of this section is

9.1. Lemma. *If q is in some Q_δ, then*

$$S^{\check{}} S = I - P.$$

The proof of this lemma is based on the following three lemmas, which give a continuity property of S, $S^{\check{}}$ and P as functions of q.

9.2. Lemma. *Let S, S_0 correspond to q, $q_0 \in Q$. Then $\|S - S_0\| \to 0$ as $|q - q_0|_Q \to 0$.*

9.3. Lemma. *Let S, S_0 correspond to q, q_0 in Q_δ (some $\delta > 0$). Then $\|S^{\check{}} - S_0^{\check{}}\| \to 0$ as $|q - q_0|_Q \to 0$.*

9.4. Lemma. *Let P, P_0 correspond to q, q_0* in *Q_δ (some $\delta > 0$). Then*

$$((P - P_0)f, h) \to 0 \qquad \text{as} \quad |q - q_0|_Q \to 0$$

for each f, h in $C_c[0, \infty)$.

We now prove Lemma 9.1. Let $0 < \delta_1 < \delta$. By Lemma 5.8 we can choose $q_n \in C_c[0, \infty)$, $n = 1, 2, \ldots$, such that $q_n \in Q_{\delta_1}$ and $|q_n - q|_Q \to 0$ as $n \to \infty$. Let S_n, P_n correspond to q_n. By Lemma 8.1,

$$(S_n^{\check{}} S_n f, h) = ((I - P_n) f, h).$$

The desired conclusion now follows from the three lemmas and the fact that $C_c[0, \infty)$ is dense in $L^2(dx)$.

In the remainder of this section we let T, S, P, w, v, $\tilde{v}$, A, $\tilde{A}$, W and G correspond to q; G is the Green's function corresponding to $(T - \lambda)^{-1}$. We let T_0, S_0, $P_0, \ldots$ be the corresponding quantities for q_0.

We now prove Lemma 9.2. Now $Sf(s) = \int w(s, x) f(x)\, dx$ where $w(s, x) = \tilde{A}(s)\, e^{isx} v(s, x) - A(s)\, e^{-isx} \tilde{v}(s, x)$. We consider the first term in w and w_0. The second terms are treated similarly. So,

$$\int [\tilde{A}(s)\, e^{isx} v(s, x) - \tilde{A}_0(s)\, e^{isx} v_0(s, x)]\, f(x)\, dx \tag{1}$$

$$= \tilde{A}(s) \int e^{isx} (v(s, x) - v_0(s, x)) f(x)\, dx$$

$$+ (\tilde{A}(s) + \tilde{A}_0(s)) \int e^{isx} v_0(s,\ x)\, f(x)\, dx.$$

Using the Plancherel theorem, the second term on the right-hand side of (1) has L^2 norm $\leqslant$

$$\sup_{s > 0} |\tilde{A}(s) - \tilde{A}_0(s)| \left[\left(\int |v_0(s, x) - 1|^2\, ds\, dx \right)^{1/2} \|f\|_{L^2} + \|f\|_{L^2} \right],$$

and this $\to 0$ as $n \to \infty$ by Lemma 5.7 ($\tilde{A}(s) = \tilde{v}(s, 0)$) and Lemma 5.6. Now for the first term, by Lemma 5.7, $\sup |\tilde{A}(s)|$ $(s > 0)$ is bounded as $|q - q_0|_Q \to 0$, so it suffices to show that

$$\int |v(s, x) - v_0(s, x)|^2\, ds\, dx \to 0 \tag{2}$$

as $|q - q_0|_Q \to 0$. For any $x, s > 0$,

$$|v(s, x) - v_0(s, x)| \leqslant \int_x^\infty |K_s(x, t)(q(t) - q_0(t))|\, dt\, \|v\|_\infty$$

$$+ \int_x^\infty |K_s(x, t)\, q_0(t)|\, dt\, \|v - v_0\|_\infty$$

where $\|v\|_\infty = \sup |v(s,x)|$ $(s, x > 0)$ and

$$K_s(x,t) = (e^{i2s(t-x)} - 1)/i2s.$$

Thus, (2) follows from (4.1) and Lemma 5.7. Lemma 9.2 is now proved.

We now prove Lemma 9.3. Recall that

$$S^{\check{}} g(s) = \int_0^\infty w(s,x)\, g(s)\, W(s)\, ds$$

where $W(s) = -1/2\pi A(s)\tilde{A}(s)$. Then we can express $(S^{\check{}} - S_0^{\check{}})g$ as

$$\int (w - w_0)\, Wg\, ds + \int w_0(W - W_0)\, g\, ds.$$

The first integral is treated as in the proof of Lemma 9.2 except that we use the fact that W remains bounded as $|q - q_0|_Q \to 0$ because of Lemma 5.8. The L^2 norm of the second integral above $\to 0$ as $|q - q_0|_Q \to 0$ since $S_0^{\check{}}$ is bounded (see Lemma 6.2) and $\sup_{s>0} |W - W_0| \to 0$ because of Lemma 5.7.

We now prove Lemma 9.4. Let γ be a simple closed curve which is disjoint from $[0, \infty)$ and which encloses σ_0 in the complex variable sense. By Lemma 2.6, σ will be inside γ for $|q - q_0|_Q$ sufficiently small. Since

$$P = (1/2\pi i) \int_\gamma (\lambda - T)^{-1}\, d\lambda$$

and similarly for P_0, it suffices to show that for each $b > 0$,

$$|G(\lambda, x, t) - G_0(\lambda, x, t)| \to 0$$

as $|q - q_0|_Q \to 0$ uniformly for $\lambda \in \gamma$ and for $0 \leqslant x,\ t \leqslant b$. We consider $\lambda = s^2 \in \gamma$ where $\operatorname{Im} s > 0$ (the other case is handled similarly). The Green's function corresponding to $(T - \lambda)^{-1}$ is given by

$$G(\lambda, x, t) = C(\lambda) \left| \begin{array}{ll} v_1(\lambda, x)\, v_2(\lambda, t), & x \leqslant t, \\ v_1(\lambda, t)\, v_2(\lambda, x), & x > t, \end{array} \right.$$

where v_1, v_2 solve $-u'' + qu = \lambda u$, $v_1(\lambda, 0) = 0$, $v_1'(\lambda, 0) = 1$ and $v_2(\lambda, x) = y(s, x) \sim e^{isx}$ and $C(\lambda) = 1/y(s, 0)$ since we can evaluate the Wronskian at $t = 0$. Let C_0, v_{10}, v_{20} denote the corresponding functions in the Green's function of $(T_0 - \lambda)^{-1}$. By Lemma 5.7, $|C(\lambda) - C_0(\lambda)| \to 0$ uniformly for $\lambda \in \gamma$ as $|q - q_0|_Q \to 0$. We now show $|v_1(\lambda, x) - v_{10}(\lambda, x)| \to 0$ as

$|q-q_0|_Q \to 0$ uniformly in λ and x. From the variation of parameters formula, $u=v_1$ solves the equation

$$u(s,x)=s^{-1}\sin(sx)+\int_0^x [s^{-1}\sin s(x-t)]\, q(t)\, u(t)\, dt,$$

and $u=v_{10}$ solves this equation with q replaced by q_0. Let $V_{s,q}$ denote the operator on $C[0,b]$ defined by

$$V_{s,q}u(x)=\int_0^x [s^{-1}\sin s(x-t)]\, q(t)\, u(t)\, dt.$$

Then

$$v_1=(I-V_{s,q})^{-1}s^{-1}\sin(sx).$$

Since $V_{s,q}$ is quasi-nilpotent and a continuous function of s and q (for q in Q), v_1 is bounded in $C[0,b]$ for $s^2\in\gamma$ and as $|q-q_0|_Q\to 0$. But,

$$v_1-v_{10}=(I-V_{s,q_0})^{-1}(V_{s,q}-V_{s,q_0})\, v_1,$$

so the desired conclusion follows. Finally, since $v_2=y=e^{isx}v(s,x)$, it follows from Lemma 5.7 that

$$|v_2(s,x)-v_{20}(s,x)|\to 0$$

as $|q-q_0|_Q\to 0$ uniformly in s, x.

10. S IS ONTO

Throughout this section we assume that q is in some Q_δ. The main purpose of this section is to prove Lemmas 10.1 and 10.2. See Sections 1 and 2 for basic notation.

10.1. LEMMA. *Suppose q is in some Q_δ. Then the map $S: H_1 \to L^2$ is one-to-one, onto and bicontinuous; and, its inverse is $S^{\vee}$.*

The following lemma generalizes Lemma 7.2 by removing the assumption that q is bounded.

10.2. LEMMA. *Suppose q is in some Q_δ. If $g\in D_M$, then*

(i) $S^{\vee}g\in D_T$,

(ii) $S^{\vee}Mg=TS^{\vee}g$,

(iii) $\|S^{\vee}g\|_{D_T}\leqslant \|S^{\vee}\|\,\|g\|_{D_M}$.

We will need the following three lemmas, which we prove at the end of this section.

10.3. Lemma. *Suppose that* $\lambda \notin \sigma_0 \cup [0, \infty)$. *If* $q \in Q$, *then*

(i) $S(\lambda - T)^{-1}f = (\lambda - M)^{-1}Sf, f \in L^2$;

(ii) $S\check{}(\lambda - M)^{-1}g = (\lambda - T)^{-1}S\check{}g, g \in L^2$.

10.4. Lemma. *Suppose that* Y *is a subspace of* $L^2(ds)$ *such that*:

(i) Y *is closed*;

(ii) $(\lambda - s^2)^{-1}Y \subset Y$ *for each* $\lambda \notin \sigma_0 \cup [0, \infty)$ *where* σ_0 *is a finite set*;

(iii) *for each finite closed interval* $J \subset (0, \infty)$, *there corresponds an* h *in* Y *such that* $|h| \geqslant \delta > 0$ *a.e. on* J.

Then $Y = L^2(ds)$.

10.5. Lemma. *If* $\int |q|\, dx < \infty$, *then* $T_q^* = T_{\bar{q}}$.

We now prove Lemma 10.1. By Lemma 9.1, $S\check{}S = I - P$. In particular,

$$(1) \qquad S\check{}Sf = f, \qquad f \in H_1.$$

Thus, $S \mid H_1$ is one-to-one with inverse $S\check{}$; and, since S, $S\check{}$ are bounded (Section 6), S is bicontinuous. It remains to show that $SH_1 = L^2(ds)$. Let $Y = SH_1$. To prove that $Y = L^2(ds)$, it suffices to show (i)–(iii) of Lemma 10.4. Since S is bicontinuous, Y is closed.

We now verify (ii). Let $Y_1 = \{g\colon g \in L^2(ds), S\check{}g \in H_1\}$ and $N = \{g\colon g \in L^2(ds), S\check{}g = 0\}$. Clearly $Y_1 = Y \oplus N$ and

$$(1) \qquad \text{for } g \in Y_1, \qquad g \in Y \Leftrightarrow g = SS\check{}g,$$

using (1). Now let $g \in Y$. Then by Lemma 10.3, $S\check{}(\lambda - M)^{-1}g = (\lambda - T)^{-1}S\check{}g \in (\lambda - T)^{-1}H_1 \subset H_1$ where the latter containment holds because P and $(\lambda - T)^{-1}$ commute. Thus, $(\lambda - M)^{-1}g \in Y_1$. But, by Lemma 10.3,

$$SS\check{}(\lambda - M)^{-1}g = (\lambda - M)^{-1}g.$$

Thus, by (1), $(\lambda - M)^{-1}g \in Y$. So, (ii) is verified.

We now verify (iii). Let J be a closed finite interval $\subset (0, \infty)$. Let f be the characteristic function of $0 \leqslant x \leqslant \delta$ for some $\delta > 0$, which we determine below. For s in J, $w'(s, 0) \neq 0$ ($' = d/dx$) since $w(s, 0) = 0$. So,

$$w(s, x) = w'(s, 0)x + w''(s, \xi)\, x^2/2$$

for $0 < \xi < x$ where ξ depends on s, x. Thus,

$$\left|\int_0^\delta w(s,x)\,dx - w'(s,0)\,\delta^2/2\right| \leqslant C\delta^3/b$$

where $|w(s,x)| \leqslant C$ for $s \in J$, $0 \leqslant x \leqslant 1$ (assume $\delta \leqslant 1$) by Lemma 5.5. Also by Lemma 5.5, $|w'(s,0)| \geqslant \delta_1 > 0$ for $s \in J$. Thus, if we choose δ so that $C\delta^3/b < \delta_1\,\delta^2/4$, then

$$\left|\int_0^\delta w(s,x)\,ds\right| \geqslant \delta_1\,\delta^2/4, \qquad s \in J.$$

Thus, $|Sf| \geqslant \delta_1\,\delta^2/4$ on J. Now let $f_1 = (I-P)f$, so that $f_1 \in H_1$. To finish the proof of (iii), it suffices to show that $Sf_1 = Sf$; hence, it suffices to show

$$SP = 0. \tag{2}$$

For $\lambda \notin \sigma(T)$,

$$SH_0 = S(\lambda - T)^{-1}H_0 = (\lambda - M)^{-1}SH_0$$

by Lemma 10.3 and the fact that P commutes with $(\lambda - T)^{-1}$. Also, H_0 is finite dimensional by Lemma 2.4. Since SH_0 is finite dimensional and invariant under each $(\lambda - M)^{-1}$, $SH_0 = \{0\}$. This completes the proof of Lemma 10.1.

We now prove Lemma 10.2. Suppose $0 < \delta_1 < \delta$. For $n = 1, 2, \ldots$, choose $q_n \in Q_{\delta_1}$ such that q_n is bounded, $q_n \to q$ locally uniformly and $|q - q_n|_Q \to 0$ as $n \to \infty$. Let $\check{S}$, $T = T_q$ correspond to q and $\check{S}_n$, $T_n = T_{q_n}$ correspond to q_n. In particular, by Lemma 9.3, $\|\check{S}_n - \check{S}\| \to 0$ as $n \to \infty$. Let $g \in D_M$. Since each q_n is bounded, by Lemma 7.2, $\check{S}_n g \in D_{T_n}$ and $T_n \check{S}_n g = \check{S}_n Mg$. Let $f_n \equiv \check{S}_n g$, $f \equiv \check{S} g$ and $k \equiv \check{S} Mg$. Since $\|\check{S}_n - \check{S}\| \to 0$ and $T_n \check{S}_n g = \check{S}_n Mg$, we know that $f_n \to f$ and $T_n f_n \to k$ as $k \to \infty$. Let Y denote

$$\{u : u \in C_c^{(2)}[0,\infty),\, u(0) = 0\}$$

and let $\lambda \notin \sigma(T)$. We first note that

$$((T_n - \lambda)f_n, h) \to (k - \lambda f, h) \qquad \text{as} \quad n \to \infty. \tag{1}$$

By Lemma 10.5, $D_{T_n^*} = D_{T_{\bar{q}_n}} \supset Y$, so

$$((T_n - \lambda)f_n, h) = (f_n, T_n^* h) - (f_n, -\bar{\lambda} h). \tag{2}$$

Now

$$(f, T^*h) - (f_n, T_n^* h) = (f - f_n, T^*h) + (f_n, T^*h - T_n^*h) \to 0 \tag{3}$$

as $n \to \infty$, because $\|f - f_n\| \to 0$, $\|f_n\|$ is bounded as $n \to \infty$ and

$$(f_n, T^*h - T_n^*h) = \int f_n(q - q_n)\bar{h}\, dx \to 0$$

as $n \to \infty$ since $q_n \to q$ locally uniformly and $h \in C_c^{(2)}[0, \infty)$. From (1), (2) and (3) we have

$$(4) \qquad (f, (T^* - \bar{\lambda})h) = (k - \lambda f, h).$$

However,

$$(5) \qquad (k - \lambda f, h) = ((T - \lambda)^{-1}(k - \lambda f), (T - \lambda)^* h).$$

Since $\sigma(T^*) = \overline{\sigma(T)}$, $\bar{\lambda} \notin \sigma(T^*)$ and by Lemma 10.5, $T_q^* = T_{\bar{q}}$; thus, $(T_q^* - \bar{\lambda})Y = (T_{\bar{q}} - \bar{\lambda})Y$, which is dense since $T_{\bar{q}}$ is defined as the closure of the operator $T_{\bar{q}} | Y$. From this and (4) and (5), we conclude that

$$f = (T - \lambda)^{-1}(k - \lambda f).$$

Thus, $f \in D_T$ and $Tf = k$ or $S^{\smile} g \in D_T$ and $TS^{\smile} g = S^{\smile} Mg$, which proves (i) and (ii) of the lemma. The inequality (iii) follows from (iii) in the case of S_n, (ii), the definition of the domain norm and the fact that $\|S_n^{\smile} - S^{\smile}\| \to 0$.

For Lemma 10.3, (i) follows easily from Lemma 7.1 and Lemma 2.3 and (ii) follows easily from Lemma 10.2 and Lemma 2.3.

We now prove Lemma 10.4. It suffices to show that $\chi_J \in Y$ where J is a closed finite subinterval of $(0, \infty)$ and χ_J is its characteristic function. Choose $f \in Y$ so that $|f| \geqslant \delta > 0$ on J. Let C_0 denote the continuous complex-valued functions on $[0, \infty)$ which tend to 0 at infinity. We know from page 246 of [1] that C_0 is the closed (uniform norm) linear span of the set of functions

$$\{(\lambda - s^2)^{-1} : \lambda \notin \sigma_0 \cup [0, \infty)\}.$$

Thus, $C_0 Y \subset Y$. Since $(1/f)\chi_J$ is the pointwise limit (a.e.) of a bounded sequence in C_0, $\chi_J = (1/f)\chi_J f \in Y$.

We now prove Lemma 10.5. We first show that $T_q^* \supset T_{\bar{q}}$; that is, if $f \in D_{t_{\bar{q}}}$, then $f \in D_{T_q^*}$ and $T_q^* f = T_{\bar{q}} f$. Let Y denote

$$\{u : u \in C_c^{(2)}[0, \infty), u(0) = 0\}.$$

From (2.4) we see that if $f, h \in Y$, then $(Tf, h) = (f, T_{\bar{q}} h)$. This shows that $h \in D_{T_q^*}$ and $T_q^* h = T_{\bar{q}} h$. From the definition of $T_{\bar{q}}$ and the fact that T_q^* is closed, we have $T_q^* \supset T_{\bar{q}}$. Now suppose that $g \in D_{T_q^*}$ and $f \in D_{T_q}$. By

Lemma 2.3 we can choose λ so that $\lambda \notin \sigma(T_q)$ and $\bar{\lambda} \notin \sigma(T_{\bar{q}})$. Let u be the solution of $(T_{\bar{q}} - \bar{\lambda})u = (T_q^* - \bar{\lambda})\, g$. Then

$$\begin{aligned}((T_q - \lambda)f, g) &= (f, (T_q^* - \bar{\lambda})\, g) = (f, (T_{\bar{q}} - \bar{\lambda})u) \\ &= ((T_q - \lambda)f, u).\end{aligned}$$

Thus, $g = u \in D_{T_{\bar{q}}}$.

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